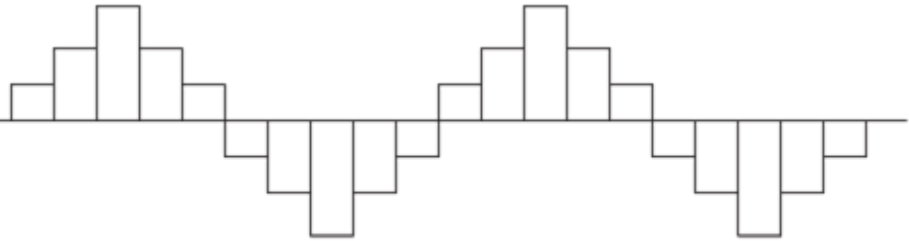
CSE1101 Discrete Mathematics for Computer Science



Relations and Functions

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Original Slides by Alicia Layne

Binary Relations

Ordered Pair

(x, y)

x = first coordinate

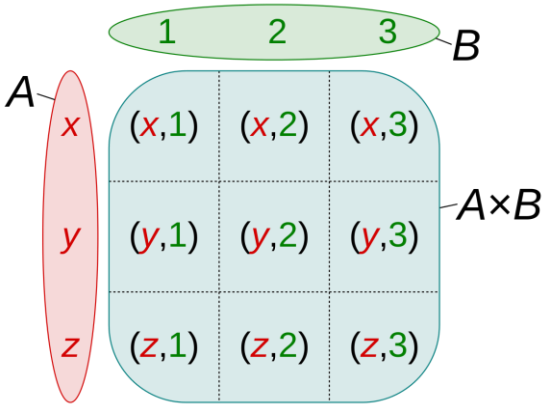
y = second coordinate

- A and B are two sets.
- A relationship between elements of A and B can be represented by a collection of **ordered pairs** – defined as the *Cartesian product* of A and B
- Since this is a relation between two sets, it is called a **binary** relation
- For a binary relation R we have $R \subseteq A \times B$

Cartesian Product

$$\boldsymbol{A \times B = \{(x, y): x \in A, y \in B\}}$$

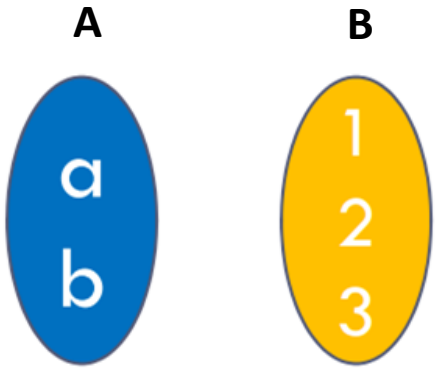
- The Cartesian Product of two sets A and B is the set of all Ordered Pairs (x, y)
  - where the first element of ordered pairs "x" belongs to first set "A"
  - and second element of ordered pairs "y" belongs or second set "B"



Cartesian Product

- **Example:**

$Consider\ Set\ A = \{a, b\}\ and\ Set\ B = \{1, 2, 3\}$



$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

Any subset of $A \times B$ is a relation R from A to B

Cartesian Product

Any subset of $A \times B$ is a relation R from A to B

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

- …

Set A = {a, b} (m) and Set B = {1, 2, 3} (n)

Total number of relation possibilities $= 2^{m \times n}$

$= 2^{2 \times 3} = 64$ possibilities

Cartesian Product

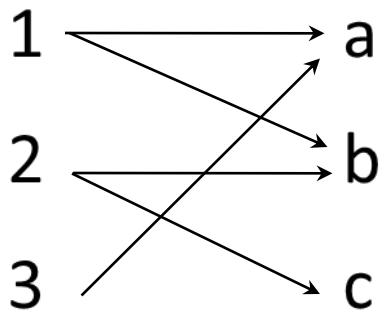
$Given\ that\ A = \{1, 3\}\ and\ B = \{2, 4, 6\},$
$find\ A \times B$

$A \times B = \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6)\}$

Representing Relations

- $Let\ A = \{1, 2, 3\}\ and\ B = \{a, c, b\}$
  $R = \{(1, a), (1, b), (2, b), (2, c), (3, a)\}$

  where R is a relation from A to B



R	a	b	c
1	x	x	
2		x	x
3	x		

Example

Let $A = \{1, 2, 3\}$ and $R = \{(a, b) \mid a \leq b\}$

What are the elements of R?

Note R is a relation from A to A.

$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\} A = \{1, 2, 3\}$ and $A = \{1, 2, 3\}$

Properties of Relations: Reflexive

- A relation R on a set A is reflexive if $(a, a) \in R$ for every element
-
  - For every element of a that belongs to set A, each element is related to itself by some relation R
- Example:

This relation is reflexive because ordered pairs $(1,1)$, $(2,2)$ and $(3,3)$ are in the relation

Properties of Relations: Reflexive

Are the following relations on A = {1, 2, 3} reflexive?

- R1 = {(1, 1), (2, 2), (3, 3)} YES
- R2 = {(1, 1), (1, 2), (3, 2), (3, 3)} NO
- R3 = {(1, 1), (1, 2), (3, 1), (2, 2)} NO
- R4 = {(1, 1), (2, 2), (3, 3), (1, 2), (2,1)} YES

Properties of Relations: Symmetric

- A relation R on a set A is symmetric if
  $\forall\, a, b \in A\, [\, (a, b) \in R \rightarrow (b, a) \in R\,]$

  For every a and b in A, if (a, b) is in R, then (b, a) is in R

- Example:

  $A = \{1, 2, 3, 4\}$

  $R = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1),$
  $(3,2), (3,4), (4,1), (4,2), (4,3)\}$

  This relation is symmetric because $(a, b) \in R$ and $(b, a) \in R$

Properties of Relations: Symmetric

Are the following relations on $A = \{1, 2, 3\}$ symmetric?

- $R1 = \{(1,1), (1,2), (2,1)\}$ YES
- $R2 = \{(1, 1), (2, 2), (3, 3)\}$ YES
- $R3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$ YES
- $R4 = \{(3, 1), (1, 3), (2, 3)\}$ NO

Properties of Relations: Transitive

- A relation R on a set A is transitive if
  $\forall\, a, b, c \in A\, [\, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R\,]$
  i.e., if $a\ R\ b, b\ R\ c \Rightarrow a\ R\ c$
- Example:
  $R4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$ is transitive

  Because:
  (a, b) AND (b, c) THEN (a, c)
  $(4,2)$ AND $(2,1)$ THEN $(4,1)$

  $\boldsymbol{a = 4, b = 2, c = 1}$

Equivalence Relation

- A relation on a set A is an equivalence relation if it is *reflexive*, *symmetric* and *transitive*

Combining Relations

- Since relations from A to B are subset of $A \times B$, two relations from A to B can be combined through set operations
- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations

  $R_1 = \{(1,1), (2,2), (3,3)\}$ and

  $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain:

  $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

  $R_1 \cap R_2 = \{(1,1)\}$

  $R_1 - R_2 = \{(2,2), (3,3)\}$

  $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Exercise

- Determine whether the relation R on the set of all Web pages is reflexive, symmetric, and/ or transitive, where $(a, b) \in R$ if and only if:
  - Everyone who has visited Web page a has also visited Web page b
  - There are no common links found on both Web page a and Web page b
  - There is at least one common link on Web page a and Web page b.

Closure of a Relation

- Consider constructing a new relation r^+ from an existing relation r where r^+ contains r and satisfies some relation property P
- The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P
- Essentially, adding the missing ordered pairs to make the relation meet the given property

Reflexive Closure

- The reflexive closure of a binary relation R on a set X is the smallest reflexive relation on X that contains R
- The reflexive closure *S* of a relation *R* on a set *A* is given by

  $S = R \cup \{(x, x\} \mid x \in A \}$: the reflexive closure of R is the union of R with the identity relation on A
- The reflexive closure of a relation R on A is obtained by adding (a, a) to R for each $a \in A$

Reflexive Closure

- Example:

  Let A = {1, 2, 3, 4} and the relation R on A

  R = {(1, 1), (2, 2), (4, 4)}

  Then by the definition of a reflexive closure:

  $S = R \cup \{(x, x) \mid x \in A\} = \{(1, 1), (2, 2), (4, 4), (3, 3)\}$

Symmetric Closure

- The symmetric closure of a binary relation R on a set X is the smallest symmetric relation on X that contains R
- The symmetric closure S of a relation R on a set A is given by

  $S = R \cup \{(a,b) \mid (b,a) \in R\}$: the symmetric closure of R is the union of R with its converse relation
- The symmetric closure of R is obtained by adding (b,a) to R for each $(a,b) \in R$

Symmetric Closure

- Example:

  Let $A = \{1, 2, 3\}$ and the relation on R on A

  $R = \{(1, 2), (2, 3), (3, 2)\}$

  Then by the definition of a symmetric closure:

$$S = R \cup \{(a, b) \mid (b, a) \in R\} = \{(1, 2), (2, 3), (3, 2), (2,1)\}$$

Transitive Closure

- The transitive closure of a binary relation R on a set X is the smallest relation on X that contains R and is transitive
- The transitive closure S of a relation R on a set A is given by

  $S = R \cup \{(a, c) \mid (a, b) \in R \wedge (b, c) \in R\}$
- The transitive closure of R is obtained by repeatedly adding (a, c) to R for each $(a, b) \in R$ and $(b, c) \in R$

Transitive Closure

- Example:

  Let $A = \{1, 2, 3\}$ and the relation R on A

  $R = \{(1, 2), (2, 3)\}$

  Then by the definition of a transitive closure:

  $S = R \cup \{(a, c) \mid (a, b) \in R \wedge (b, c) \in R\} = \{(1, 2), (2, 3), (1, 3)\}$

Exercise

- Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pains $(0, 1), (1, 1), (1, 2), (2, 0), (2, 2)$, and $(3,0)$.
- Find
  - Reflexive closure of R
  - Symmetric closure of R

Functions

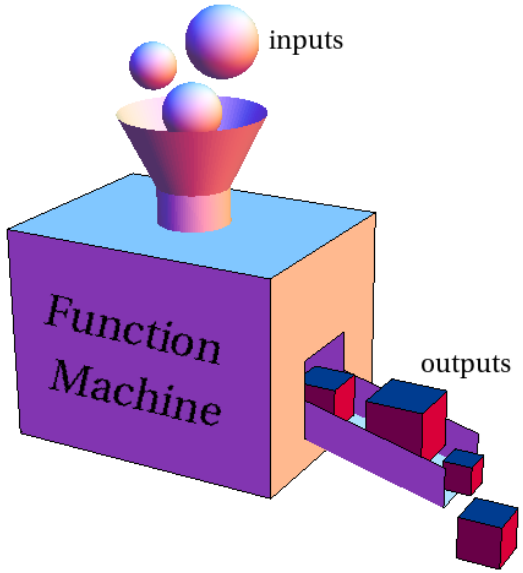
Functions

- A function (Defined as $f: X \rightarrow Y$) maps elements of one set X to elements of another set Y where X and Y are non-empty sets
- A function relates an input to an output
- X is called domain and Y is called co-domain of function 'f'
- 'x' is called pre-image and 'y' is called image of function 'f'

Input, Relationship, Output

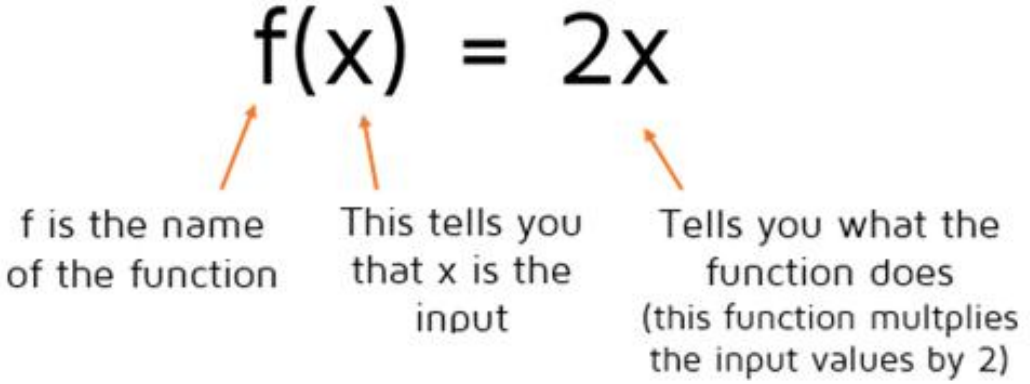
- A function relates an input to an output
- Consider $f(x) = x \times 2$

Input	Function	Output
0	× 2	0
1	× 2	2
7	× 2	14
10	× 2	20
…	…	…

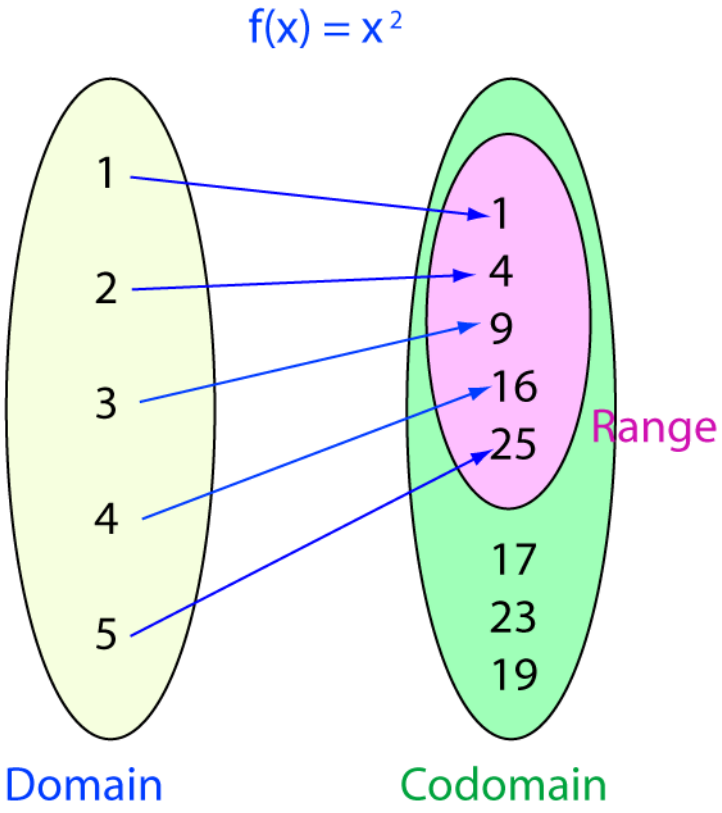


Function Notation

- Functions names and inputs names vary
- Input acts as a ‘placeholder’


f(x) = 2x
f is the name of the function
This tells you that x is the input
Tells you what the function does (this function multplies the input values by 2)

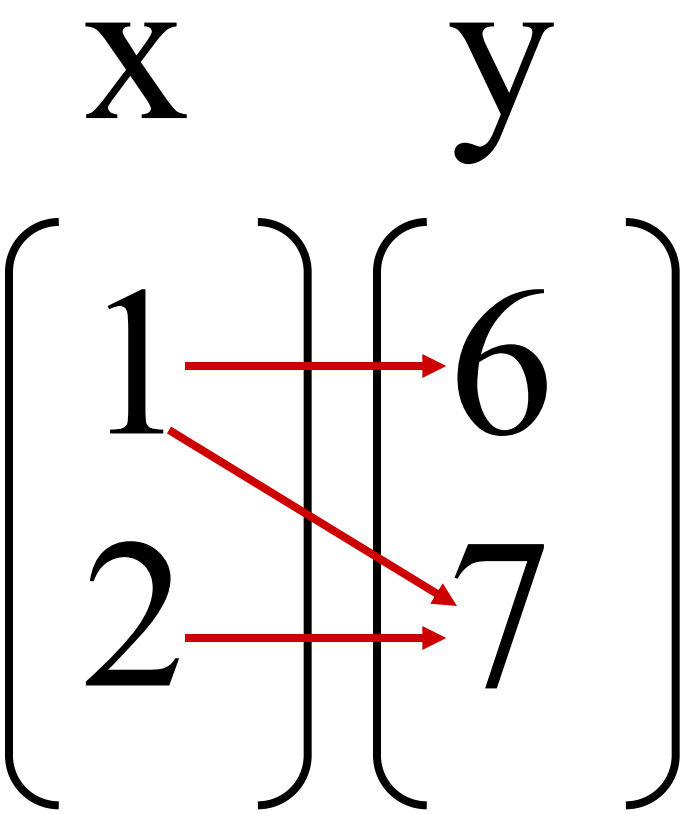
Function Terminology



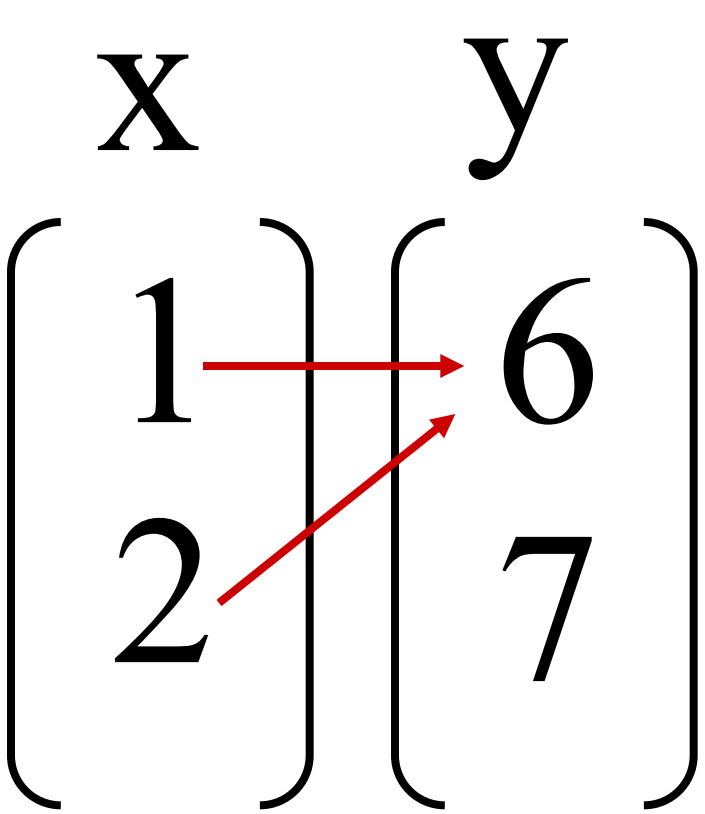
Definitions

Relation - Any set of ordered pairs

Function -A type of **relation** where there is exactly one output for every input. For every x there is exactly one y.



Not a Function



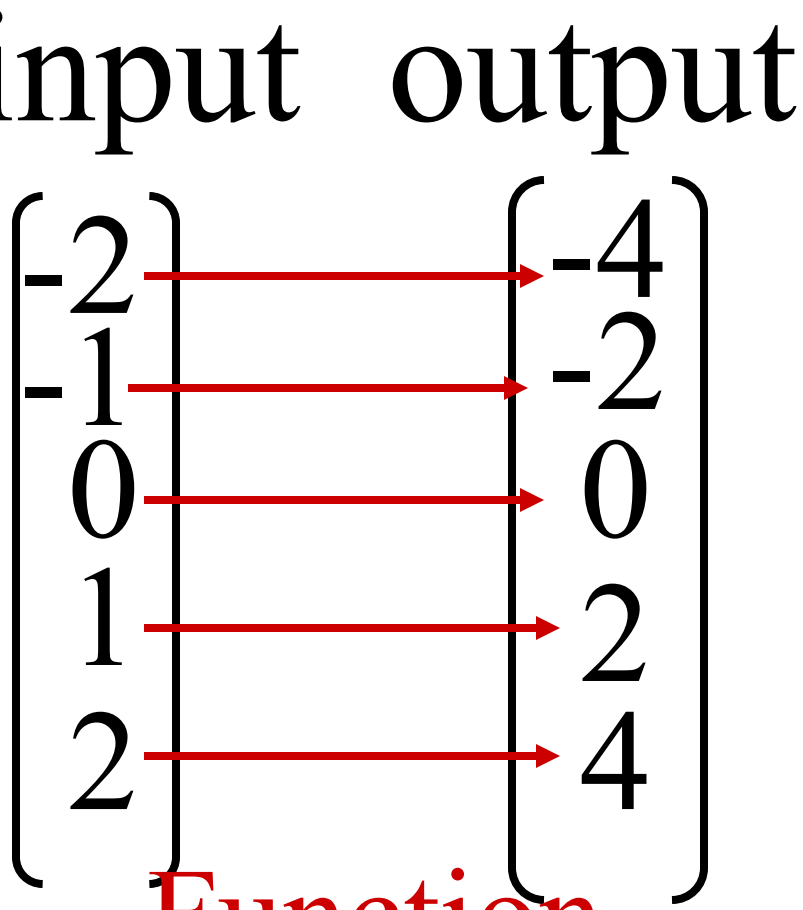
Function

y = 2x

x-y chart

x	y
-2	-4
-1	-2
0	0
1	2
2	4

mapping

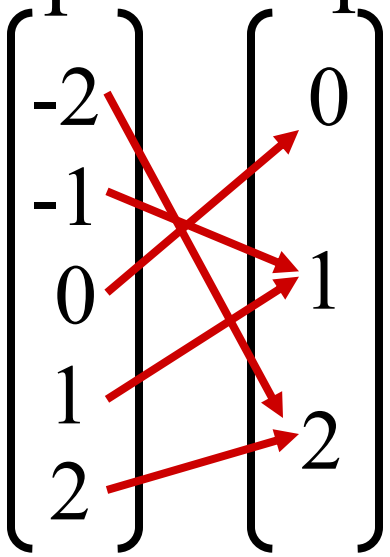


Function

Determine whether the equation is a function.

$y = |x|$

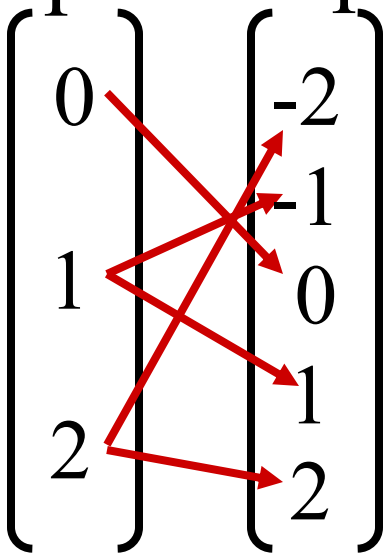
x	y
-2	2
-1	1
0	0
1	1
2	2



Function

$x = |y|$

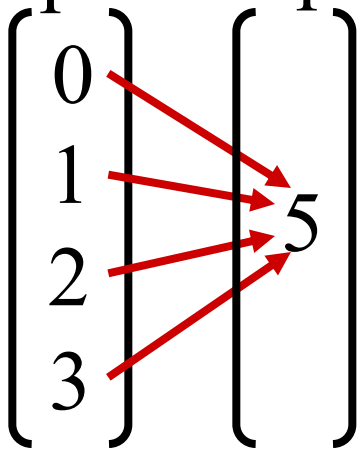
x	y
2	-2
1	-1
0	0
1	1
2	2



Not a Function

Tell whether the relation below is a function.

1) input output



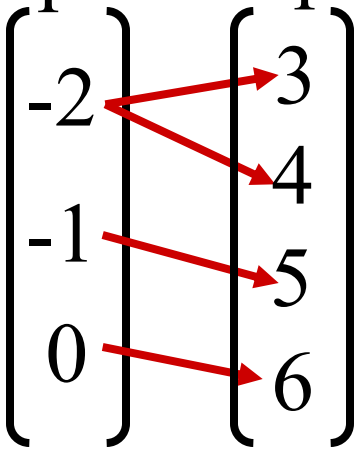
Function

2)

x	y
-3	-1
-3	0
-3	1
-3	2

Not a Function

4) input output



Not a Function

Function Notation

$y = 2x + 3$

$f(x) = 2x + 3$

when $x = 1$, $y = 5$ $f(1) = 5$

when $x = 2$, $y = 7$ $f(2) = 7$

when $x = 3$, $y = 9$ $f(3) = 9$

when $x = 4$, $y = 11$ $f(4) = 11$

$f(-4) = -5$

$g(x) = x^2$ $\quad\quad$ $h(x) = 3x - 2$

Evaluate the following.

1) $g(4) =$ 16

2) $h(-2) =$ −8

3) $g(-3) =$ +9

4) $h(5) =$ 13

5) $h(4) + g(1) =$

$10 + 1 =$ 11

6) $h(-5) \bullet g(-2) =$

$-17 \bullet 4 =$ −68

7) $g\big(h(3)\big) =$

$g(7) =$ 49

8) $h\big(g(2)\big) =$

$h(4) =$ 10

Evaluate the function over the domain, $x = -1, x = 0, x = 2$

1) $f(x) = 4x$

$\{-4, 0, 8\}$

2) $g(x) = 3x - 9$

$\{-12, -9, -3\}$

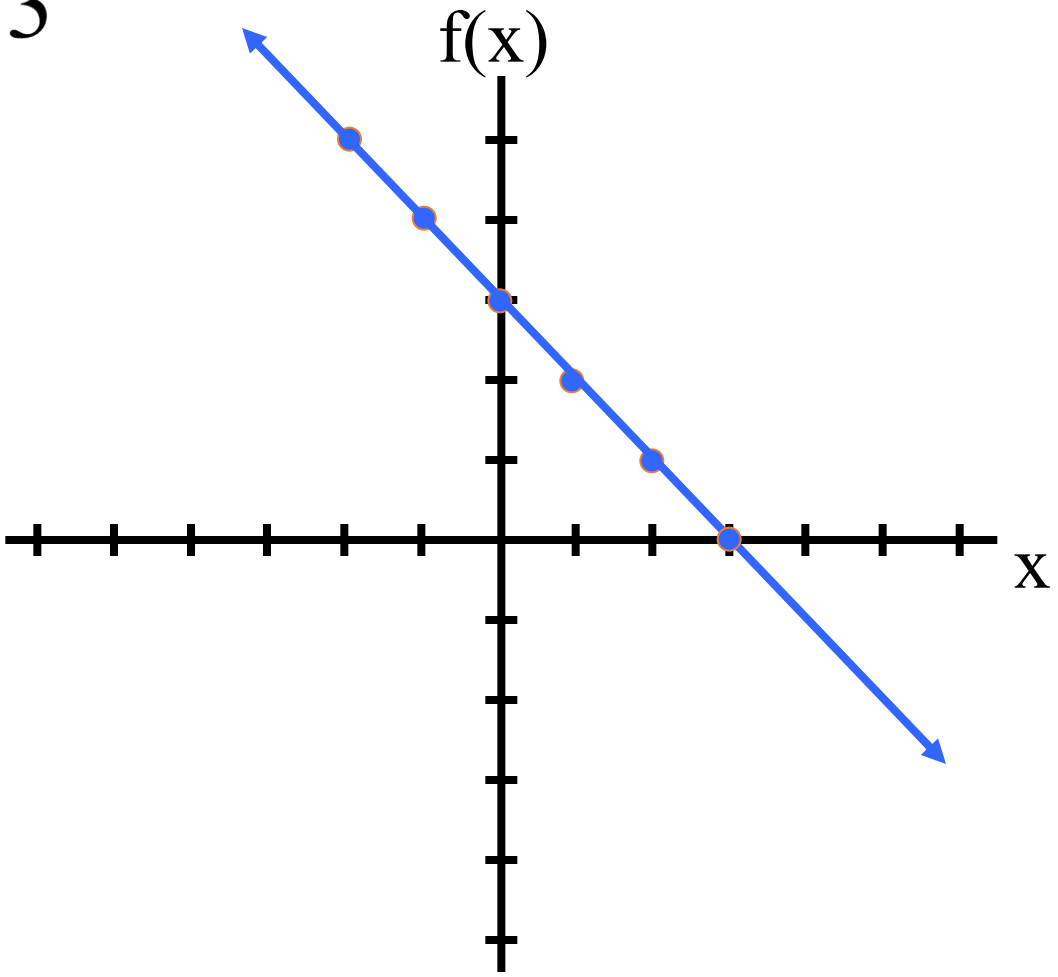
3) $h(x) = x^2 - 1$

$\{0, -1, 3\}$

Graph the linear function.

$$f(x) = -x + 3$$

x	f(x)
-3	6
-2	5
-1	4
0	3
1	2
2	1
3	0

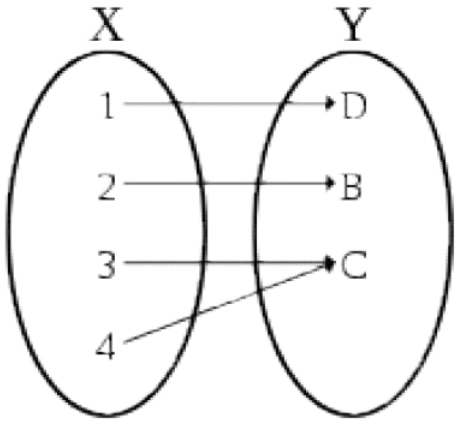


Types of Functions

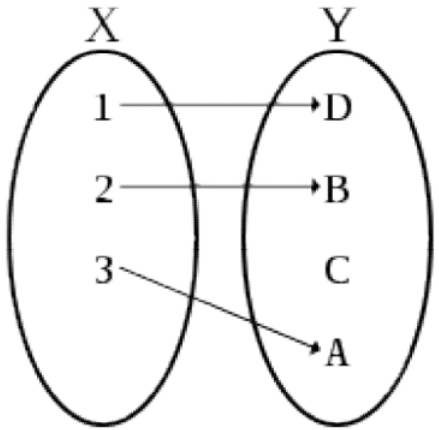
- Functions can be classified by how they map points from the domain to the codomain:
  - Injection (one-to-one)
  - Surjections (onto)
  - Bijections (one-to-one and onto)
- These properties relate to important concepts such as the inverse of a function

Injective Functions (one-to-one)

- Definition: A function *f* is **one-to-one** or injective (or an injection) if each image has at most one pre-image
- It preserves distinctness: it never maps distinct elements of its domain to the same element of its co-domain



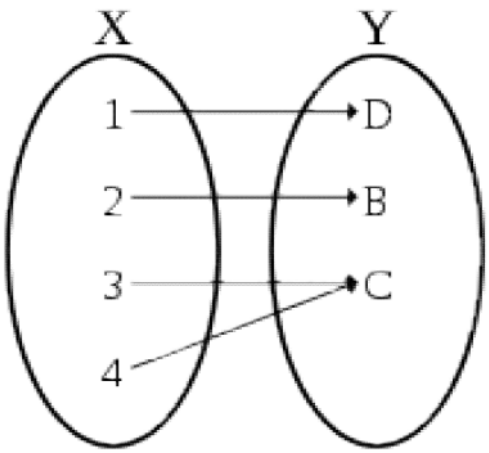
Not an injective function



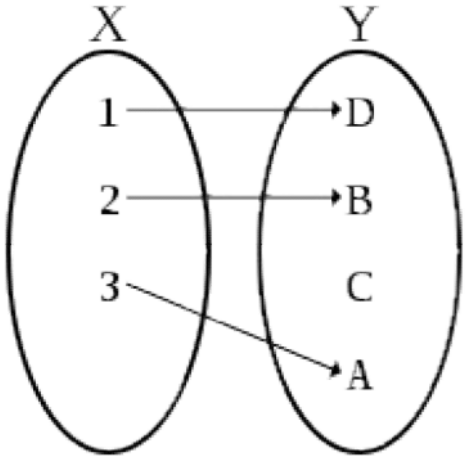
Injective function

Surjective Function (onto)

- Definition: A function is **onto** or **surjective** if every element in the co-domain is mapped
- Covers the entire domain (range == co-domain)



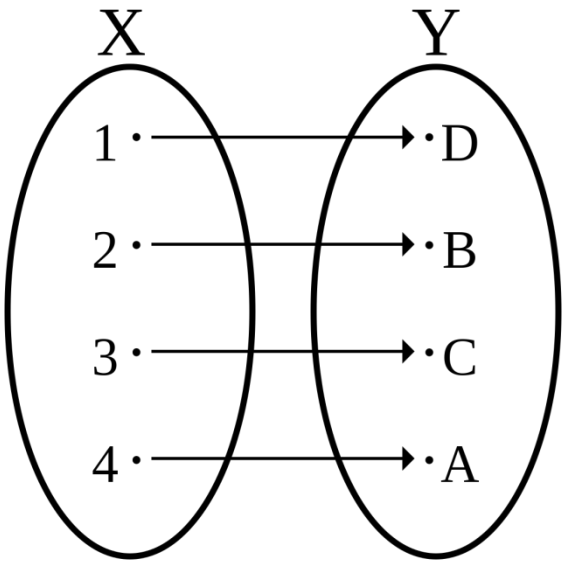
Surjective non-injective function



Not a surjective function

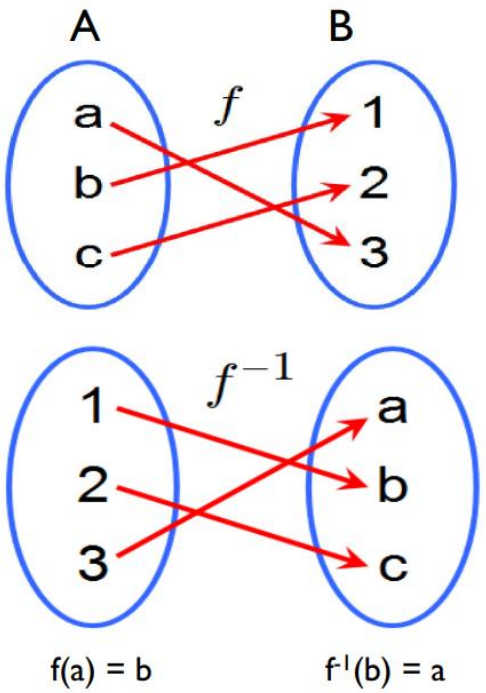
Bijective Function (one-to-one and onto)

- A function is bijective (is a bijection) if it is both an injection and surjection
- One-to-one and onto correspondence and onto



Inverse Function

- An inverse function *reverses* another function
- The inverse function of *f* is the function that assigns an element b (image) belonging to B (co-domain) to a unique element a (pre-image) in A (domain)
- If the function *f* applied to an input *x* gives a result of *y*, then applying its inverse function *g* to *y* gives the result *x*,
- The inverse function is denoted by f^{-1}

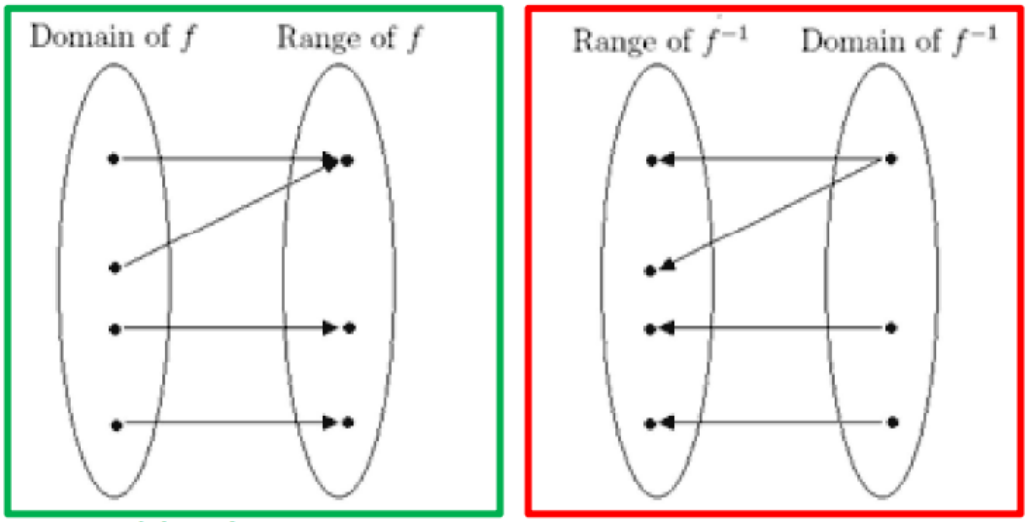


Inverse Function

- For a function f: X → Y to have an inverse, it must have the property that for every y in Y there must be one, and only one x in X so that f(x) = y
- This property ensures that a function g: Y → X will exist having the necessary relationship with f.

Inverse Function and Bijection

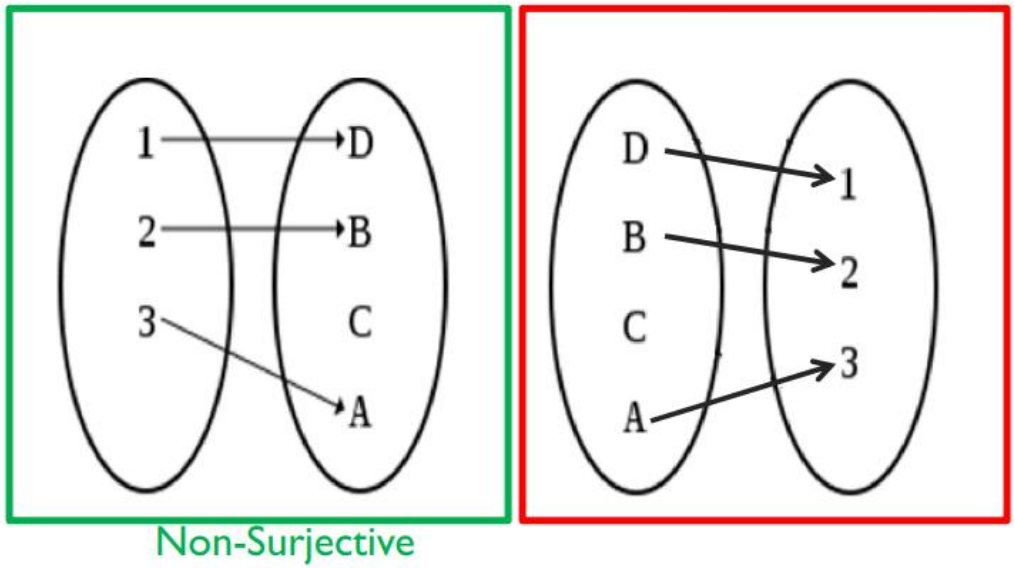
Why must a function be bijective to have an inverse?
Consider the case **where f is not one-to-one (not injective)**



How can we define the inverse? Does *f*-1(b) = a1 or a2?
Therefore, there is no inverse.

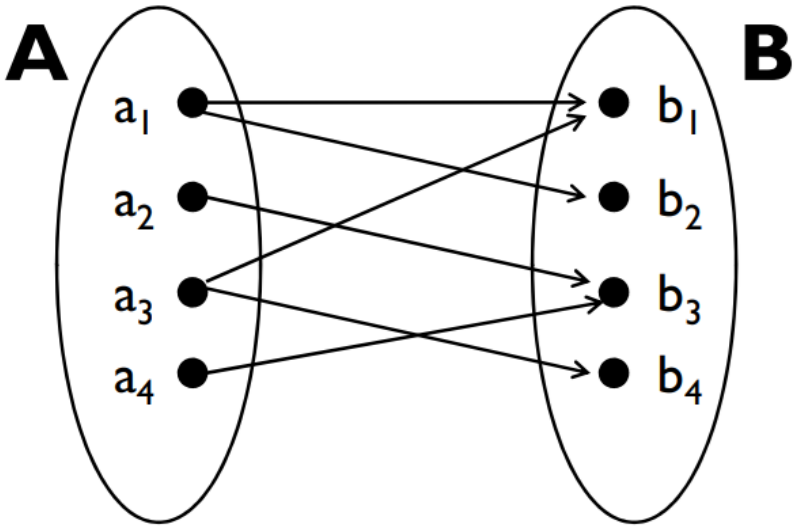
Inverse Function and Bijection

Why must a function be bijective to have an inverse?
Consider the case **where f is onto (not surjective)**



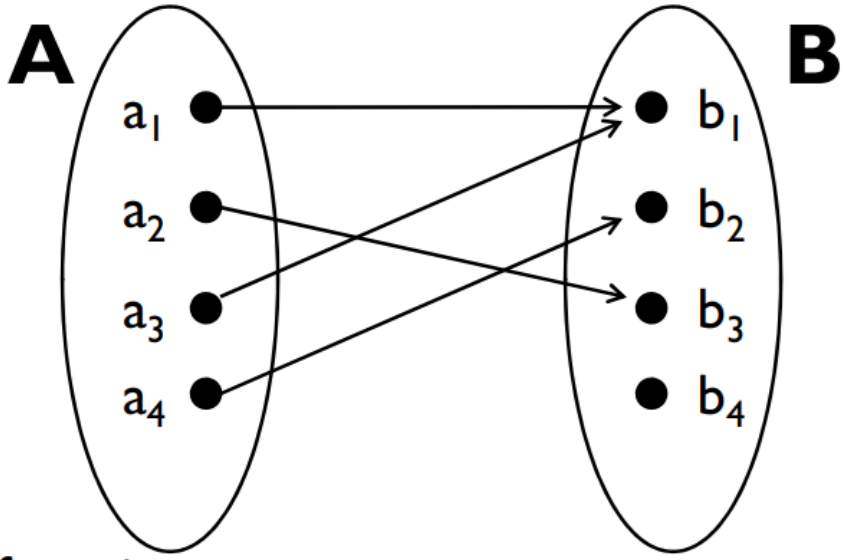
Function is undefined at C
Therefore, no inverse!

Classifying Functions Example



- Is this a function? Why?
- No, because a_1 and a_3 have two images

Classifying Functions Example

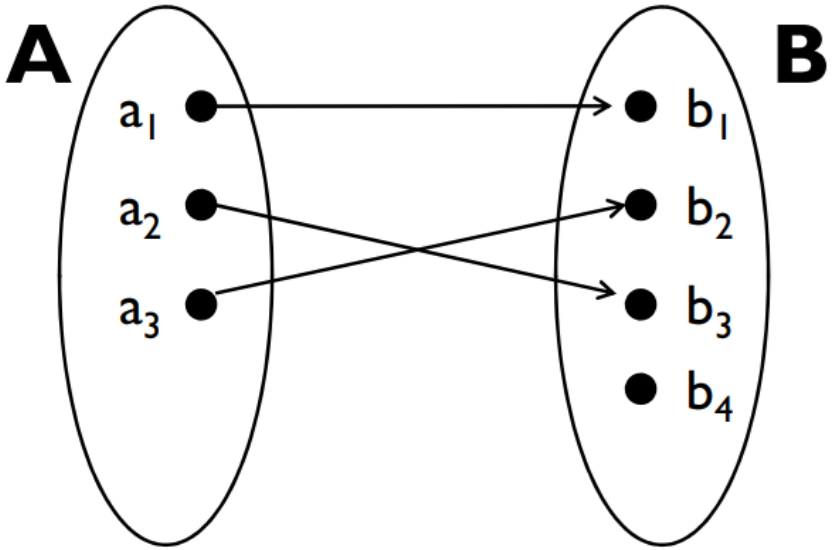


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_1 has 2 preimages

No, b_4 has no preimage

Classifying Functions Example

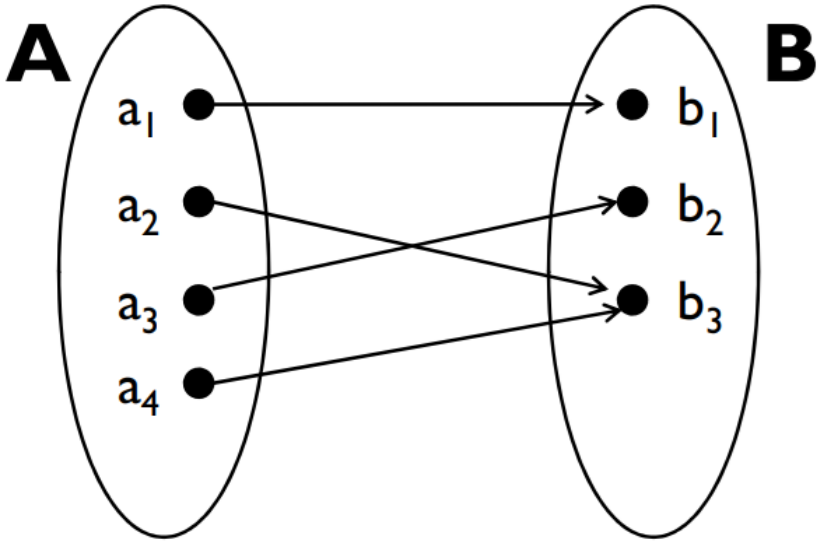


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

Yes

No, b_4 has no preimage

Classifying Functions Example

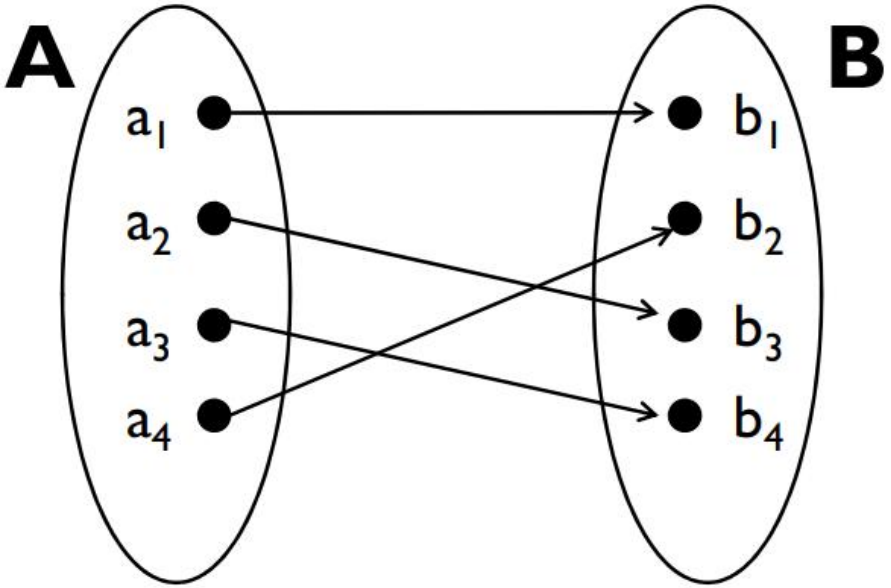


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_3 has 2 preimages

Yes, every b_i has a preimage

Classifying Functions Example



- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

It is a bijection

Arithmetic Operations with Functions

- We can perform arithmetic operations on functions. The result is a new function

  Consider two functions f and g:

  The sum of f and g

  $$(f + g)(x) = f(x) + g(x)$$

  The difference of f and g

  $$(f - g)(x) = f(x) - g(x)$$

  The product f and g

  $$(f * g)(x) = f(x) * g(x) = f(x)\, g(x)$$

  The quotient of f and g

  $$(f / g)(x) = f(x) / g(x) = f(x)/g(x)$$

Arithmetic Operations with Functions (sum)

Example

Problem

$f(x) = 5x + 6$

$g(x) = 3x^2 - 4x + 8$

Find $(f + g)(x)$.

$$(f+g)(x) = f(x) + g(x)$$
$$= (5x+6) + (3x^2 - 4x + 8)$$
$$= 3x^2 + 5x - 4x + 6 + 8$$
$$= 3x^2 + x + 14$$

Identify $f(x)$ and $g(x)$. Replace $f(x)$ with $5x+6$, and $g(x)$ with $3x^2 - 4x + 8$. Then add and combine like terms.

Answer

$(f+g)(x) = 3x^2 + x + 14$

Arithmetic Operations with Functions (difference)

Example

Problem

$f(x) = 5x + 6$

$g(x) = 3x^2 - 4x + 8$

Find $(g - f)(x)$.

$$(g - f)(x) = g(x) - f(x)$$
$$= (3x^2 - 4x + 8) - (5x + 6)$$
$$= 3x^2 - 4x + 8 - 5x - 6$$
$$= 3x^2 - 4x - 5x + 8 - 6$$
$$= 3x^2 - 9x + 2$$

Replace $g(x)$ and $f(x)$ with their respective expressions.

Then subtract and combine like terms.

Answer

$(g - f)(x) = 3x^2 - 9x + 2$

Arithmetic Operations with Functions (product)

Example

Problem

$f(x) = 2x + 1$

$g(x) = 5x - 3$

Find the product of *f* and *g*.

$(f \cdot g)(x) = f(x) \cdot g(x)$

To find the product, multiply the functions.

$(f \cdot g)(x) = (2x + 1)(5x - 3)$

$= 10x^2 - 6x + 5x - 3$

$= 10x^2 - x - 3$

Replace $f(x)$ with $(2x + 1)$, and $g(x)$ with $(5x - 3)$.

Answer

$(f \cdot g)(x) = 10x^2 - x - 3$

Arithmetic Operations with Functions (quotient)

Example

Problem

$f(x) = 12x^3 + 15x^2 - 6x$

$g(x) = 3x$

Find $\left(\frac{f}{g}\right)(x)$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{12x^3 + 15x^2 - 6x}{3x}, \quad x \neq 0$$

$$= \frac{3x(4x^2 + 5x - 2)}{3x}$$

$$= 1 \cdot (4x^2 + 5x - 2)$$

$$= 4x^2 + 5x - 2, \quad x \neq 0$$

To find the quotient, divide *f* by *g*. Substitute the polynomials in for *f*(*x*) and *g*(*x*) and divide. We add $x \neq 0$ because $x = 0$ would make the denominator $g(x) = 0$ and $\frac{f(x)}{g(x)}$ undefined. Remember to rename $\frac{3x}{3x}$ as 1.

Answer

$$\left(\frac{f}{g}\right)(x) = 4x^2 + 5x - 2$$

Composite Functions

- Function composition is an operation that takes two functions f and g and produces a new function h such that $h(x) = g(f(x))$
- Composition of functions is essentially one function is inside of another function
- The composition of functions is a special case of the composition of relations
- In composite functions the order in which the composition of the functions is written is important
  - $(f \circ g)(x) = f(g(x))$, the g function is inside of the f function
  - $(g \circ f)(x) = g(f(x))$, the f function is inside of the g function

Composite Functions

- How to find the composition of two functions:
  - Rewrite the composition in a different form
  - Replace each occurrence of x found in the outside function with the inside function
  - Simplify the answer.

Composite Functions - Example

If $f(x) = -4x + 9$ and $g(x) = 2x - 7$, find $(f \circ g)(x)$

$(f \circ g)(x) = f(g(x))$ Rewrite the composition in a different form

$= -4(2x - 7) + 9$ Replace each occurrence of x in f(x) with $g(x) = 2x - 7$

$= -8x + 28 + 9$ Simplify ...

$= -8x + 37$

Composite Functions - Example

If $f(x) = -4x + 9$ and $g(x) = 2x - 7$, find $(g \circ f)(x)$

$(g \circ f)(x) = g(f(x))$ Rewrite the composition in a different form

$= 2(-4x + 9) - 7$ Replace each occurrence of x in g(x) with f(x)

$= -8x + 18 - 7$ Simplify ...

$= -8x + 11$